1. If f is bounded on $[a,b]$ and for any $c \in (a,b)$ $f|_{[c,b]}$ is Riemann integrable, then f is Riemann integrable on $[a,b]$ and $\int_c^b f \to \int_a^b f$ as $c \to a^+$.

Pf: We wish to use Squeeze Theorem, i.e., for any $\varepsilon > 0$, find $g, h \in R[a,b]$ s.t. $g < f < h$ and $\int_a^b (g-h) < \varepsilon$.

Fix $\varepsilon > 0$.

Since f is bounded, there exists $M \in \mathbb{R}$ such that $-M < f < M$.

Take $c \in (a,b)$ with $c - a < \frac{\varepsilon}{2M}$.

Let $g(x) = \begin{cases} -M, & x \in [a, c), \\ f(x), & x \in [c, b]. \end{cases}$

$h(x) = \begin{cases} M, & x \in [a, c), \\ f(x), & x \in [c, b]. \end{cases}$

Since $g = -M\chi_{[a,c)} + f|_{[c,b]}$,

g is Riemann integrable on $[a, b]$.

Similarly, h is Riemann integrable on $[a, b]$.

Since $-M < f < M$, $g \le f \le h$.

Moreover, $h - g = 2M\chi_{[a,c)}$.

Thus $\int_a^b (h-g) = 2M(c-a) < 2M\dfrac{\varepsilon}{2M} = \varepsilon$.

By Squeeze Theorem, $f \in R[a, b]$.

$$\left| \int_a^b f - \int_c^b f \right| = \left| \int_a^c f \right| \le \int_a^c |f|$$

$$\le (c-a)M \to 0 \text{ as } c \to a^+.$$

$\square$

Rmk: Boundedness is necessary.

Counter-example: $f = \begin{cases} \frac{1}{x}, & x \in (0, 1], \\ 0, & x = 0 \end{cases}$.

Since f is unbounded,

$f \notin R[0,1]$.

But $f|_{[c,1]} \in R[0,1]$.

2. If f and g are continuous on $[a,b]$ and $g \geq 0$, then there exists $c \in [a,b]$ such that $\int_a^b fg = f(c)\int_a^b g$.

Pf: Let $m = \inf_{x\in[a,b]} f(x)$ and $M = \sup_{x\in[a,b]} f(x)$.

Since f is continuous, there exists $\alpha, \beta \in [a,b]$ such that $m = f(\alpha)$ and $M = f(\beta)$.

Since $m \leq f(x) \leq M$ and $g(x) \geq 0$,

$mg(x) \leq f(x)g(x) \leq Mg(x)$ for any $x \in [a,b]$.

Thus $m\int_a^b g \leq \int_a^b fg \leq M\int_a^b g$ and $\int_a^b g \geq 0$.

- If $\int_a^b g = 0$, then $\int_a^b fg = 0$.

  $\int_a^b fg = f(c)\int_a^b g$ for any $c \in [a,b]$.

- If $\int_a^b g > 0$, then $f(\alpha) = m \le \frac{\int_a^b fg}{\int_a^b g} \le M = f(\beta)$

By Intermediate Value Theorem, since f is continuous, there exists c between α and β such that $f(c) = \frac{\int_a^b fg}{\int_a^b g}$.

Of course, $c \in [a, b]$.

$\square$

Rmk:
- The continuity of g is unnecessary. We only need $g \in R[a, b]$.
- $g \ge 0$ is necessary.

Counter-example: $f(x) = g(x) = x$ on $[-1, 1]$.

Then $\int_{-1}^{1} fg = \int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3} > 0$

and $\int_{-1}^{1} g = \int_{-1}^{1} x \, dx = 0$.

$0 < \int_{-1}^{1} fg \ne f(c) \int_{-1}^{1} g = 0$ for any $c \in [a, b]$.

3. $L(f+g) \geq L(f) + L(g)$

Pf: For any partition P_1, P_2 of $[a,b]$,

let $P = P_1 \cup P_2$.

Exercise: You can show that $L(f;P) \geq L(f;Q)$ if $P \supset Q$.

$$L(f+g) \geq L(f+g;P) = \sum_{k=1}^{n} \inf\{f(x)+g(x) : x \in [x_{k-1}, x_k]\}(x_k - x_{k-1})$$

$$\geq \sum_{k=1}^{n} \left(\inf\{f(x) : x \in [x_{k-1}, x_k]\} + \inf\{g(x) : x \in [x_{k-1}, x_k]\} \right)(x_k - x_{k-1})$$

$$= L(f;P) + L(g;P)$$

$$\geq L(f;P_1) + L(g;P_2)$$

Since P_1 is arbitrary,

$$L(f+g) \geq L(f) + L(g;P_2)$$

Since P_2 is arbitrary,

$$L(f+g) \geq L(f) + L(g).$$

$\square$

Rmk: • The inequality can be strict in some cases.

Example: $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1], \\ 0, & x \in \mathbb{Q}^c \cap [0,1]. \end{cases}$

$g(x) = \begin{cases} 0, & x \in \mathbb{Q} \cap [0,1], \\ 1, & x \in \mathbb{Q}^c \cap [0,1]. \end{cases}$

Then $f+g \equiv 1$.

Thus $L(f+g) = 1$ and $L(f) = L(g) = 0$.

- We also have $U(f+g) \leq U(f) + U(g)$.

Since $L(f) + L(g) \leq L(f+g) \leq U(f+g) \leq U(f) + U(g)$,

if $f, g \in R[a, b]$, i.e. $L(f) = U(f)$ and $L(g) = U(g)$,

then $L(f+g) = U(f+g)$, i.e., $f+g \in R[a, b]$.